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Brans-Dicke Plane Symmetric Vacuum

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Abstract

The general solution of the mass zero scalar field coupled to the gravitational field with the assumption of plane symmetry is exhibited and partially interpreted. Energy transfer from a gravitational wave to test particles is studied invariantly.

1. *Introduction*

It is well known that a spherically symmetric vacuum gravitational field is static. Therefore the radiation problem is trivial. However, Brans–Dicke theory does seem to admit time-dependent, spherically symmetric solutions. We would, therefore, be able to study the radiation problem in this model in detail.

It is unfortunate that the field equations do not seem to admit an explicit analytic solution. It turns out that plane symmetry simplifies one field equation in an essential manner so as to allow an explicit general solution. These solutions are not asymptotically flat and are therefore difficult to interpret. The interpretation problem has not been solved in general.

We remind the reader that Brans–Dicke theory is conformally equivalent to that of mass zero scalar field. The exact mathematical formulae are given by Dicke (1962).

2. *Solution*

The Einstein field equations are given by

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \tag{2.1}$$

where $\kappa = 8\pi G_0/c^4$, and G_0 is Newton's gravitational constant. For mass zero scalar fields

$$T_{\mu\nu} = \psi_{,\mu}\psi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\psi_{,\sigma}\psi_{,}{}^{\sigma}$$

$$\Box\psi = 0 \tag{2.2}$$

From equations (2.1) and (2.2) it follows that

$$R_{\mu\nu} = -\kappa\psi_{,\mu}\psi_{,\nu} \tag{2.3}$$

where $R_{\mu\nu}$ is the Ricci tensor.

Corresponding to each solution $g_{\mu\nu}$, ψ there is a solution $\bar{g}_{\mu\nu}$, ϕ of Brans–Dicke defined by

$$\bar{g}_{\mu\nu} = \frac{1}{\phi}g_{\mu\nu}$$

$$\phi = e^{i\psi\sqrt{\kappa}\sqrt{\omega+\frac{3}{2}}} \tag{2.4}$$

It is convenient to define $\sigma = i\sqrt{\kappa}\psi$. Equation (2.3) then reads

$$R_{\mu\nu} = \sigma_{,\mu}\sigma_{,\nu} \tag{2.5}$$

This is the set of equations we shall solve for plane symmetry.

We now adopt the definition and coordinates given by Taub (1956) for this particular symmetry.

The line element can be written in the form

$$(ds)^2 = e^{\omega(t,3)}(dt^2 - dz^2) - e^{\mu(t,3)}(dx^2 + dy^2) \tag{2.6}$$

where X and Y at fixed t and z describe the planes of symmetry.

We consider it slightly more convenient to work with characteristic coordinates defined by

$$u = t - z$$

$$v = t + z$$

x and y remain the same. So the line element is transformed into

$$(ds)^2 = e^{\omega(u,v)}du\,dv - e^{\mu(u,v)}(dx^2 + dy^2) \tag{2.7}$$

For consistency we assume σ to be a function of u and v only.

The field equation given by (2.5) are

$$\sigma_+{}^2 = \mu_{++} + \tfrac{1}{2}\mu_+{}^2 - \mu_+\omega_+$$

$$\sigma_-{}^2 = \mu_{--} + \tfrac{1}{2}\mu_-{}^2 - \mu_-\omega_- \tag{2.8}$$

$$\sigma_+\sigma_- = \mu_{+-} + \tfrac{1}{2}\mu_+\mu_- + \omega_{+-} \tag{2.9}$$

$$0 = (e^{\mu})_{+-} \tag{2.10}$$

$$\sigma_{+-} = -\tfrac{1}{2}(\sigma_+\mu_- + \sigma_-\mu_+) \tag{2.11}$$

where σ_+ stands for $\partial\sigma/\partial u$ and σ_- for $\partial\sigma/\partial v$. Equation (2.9) is a consequence of the remaining four equations if σ is not a constant.

For spherical symmetry in similar coordinates all field equations are the same except (2.10) which changes into

$$-\tfrac{1}{2}e^{\omega} = (e^{\mu})_{+-}$$

This coupling of μ and ω complicates the problem very much.

It follows from (2.10) that

$$e^{\mu} = f(u) + g(v) \tag{2.12}$$

where f and g are arbitrary functions of one variable. Now we see that equation (2.11) is invariant under any transformation $\bar{u} = \bar{u}(u)$ and $\bar{v} = \bar{v}(v)$, therefore we can write

$$\frac{\partial^2 \sigma}{\partial f\, \partial g} = -\frac{1}{2}\left[\frac{\partial \sigma}{\partial f}\frac{\partial \mu}{\partial g} + \frac{\partial \sigma}{\partial g}\frac{\partial \mu}{\partial f}\right]$$

so we have

$$\frac{\partial^2 \sigma}{\partial f\, \partial g} = -\frac{1}{2}\frac{1}{f+g}\left[\frac{\partial \sigma}{\partial f} + \frac{\partial \sigma}{\partial g}\right] \tag{2.13}$$

We define $\rho = \frac{1}{2}(f+g)$ and $\tau = \frac{1}{2}(g-f)$ to get

$$\frac{\partial^2 \sigma}{\partial \tau^2} = \frac{\partial^2 \sigma}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \sigma}{\partial \rho}$$

This is nothing but the two-dimensional "spherically symmetric" wave equation. Its general solution is

$$\sigma = \int_0^\infty F(\tau + \rho \cosh \lambda)\, d\lambda + \int_0^\infty G(\tau - \rho \cosh \lambda)\, d\lambda$$

where F and G are arbitrary functions of one variable. A simple way to get this result is to Fourier analyse σ in the variable τ and use appropriate integral representations of Hankel functions of the first and second kinds†

The next step is to calculate ω from equations (2.8). A simple way is to make the coordinate transformation defined by $\bar{u} = f(u)$ and $\bar{v} = g(v)$ as an auxiliary step.

The result is

$$e^{\mu} = f(u) + g(v)$$

$$e^{\omega} = a\frac{f_+ g_-}{\sqrt{(f+g)}} \exp - \left(\int (f+g)\left[\frac{\sigma_+^{\,2}}{f_+} du + \frac{\sigma_-^{\,2}}{f_-} dv\right]\right) \tag{2.14}$$

$$\sigma = \int_0^\infty F(g - f + (f+g)\cosh \lambda)\, d\lambda + \int_0^\infty G(g - f - (f+g)\cosh \lambda)\, d\lambda$$

where a is an arbitrary constant. f and g just account for necessary coordinate arbitrariness. Physical situations are governed by F and G.

† This result is probably well known. Hints can be found in the book by E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals.*

We conclude from here that any solution of the mass zero scalar field in general relativity can be generated from a solution of the linear equation (2.13) and the expressions (2.12), (2.14) for e^{μ} and e^{ω}. e^{ω} is not defined when $g(v)$ is a constant.

The following trick can be used to get a solution for this case; consider $g(v) = \lambda v$ and $a = \alpha/\lambda$, then

$$e^{\omega} = \alpha \frac{f_+}{\sqrt{f}} \exp\left(-\int f \frac{\sigma_+^2}{f_+} du\right) \tag{2.15}$$

after the limit $\lambda \to 0$ is taken. We observe that in this case $\sigma(u)$ is an arbitrary function, and α any constant. This constant can be incorporated into the indefinite integral. The next section will be dedicated to the interpretation of this special solution.

3. *Interpretation*

We shall be concerned about the physical interpretation of the space-time defined by

$$ds^2 = \alpha\, e^{\mu/2} \frac{d\mu}{du} \exp\left(-\int \frac{\sigma_+^2}{\mu_+} du\right) du\, dv - e^{\mu(u)}(dx^2 + dy^2) \tag{3.1}$$

which we have shown to be a solution of equation (2.5). $\sigma = \sigma(u)$ is an arbitrary function of u. We observe that besides the plane symmetry killing vectors, there is another light-like motion given by $\partial/\partial v$; this means we have a rigid wave traveling at the speed of light. We can form a wave packet by assuming that the first and second derivates of $\sigma(u)$ vanish outside some finite closed interval of u values. The actual value of σ is not important because only $\partial\sigma/\partial u$ appears in the line element.

It is interesting and perhaps surprising that space-time is flat for regions where $\partial\sigma/\partial u = 0$. This can be seen from the Riemann tensor given by

$$R_{\mu\nu\sigma\tau} = -2e^{-2\omega}\left(\frac{\partial\sigma}{\partial u}\right)^2 [(\xi_\mu K_\nu - \xi_\nu K_\mu)(\xi_\sigma K_\tau - \xi_\tau K_\sigma) + (\eta_\mu K_\nu - \eta_\nu K_\mu) \times (\eta_\sigma K_\tau - \eta_\tau K_\sigma)] \tag{3.2}$$

$$e^{\omega} = e^{\mu/2} \frac{d\mu}{du} \exp - \int \frac{\sigma_+^2}{\mu_+} du$$

where $K^{\mu} = \delta_-{}^{\mu}$ is the lightlike Killing vector. ξ^{μ} and η^{ν} are the normalised Killing vectors corresponding to x and y translations.

We also exhibit this result directly by a coordinate transformation defined by

$$
\begin{aligned}
e^{\mu} &= u^2 \\
\bar{u} &= u \\
\bar{v} &= 2e^{-(1/2)\int u\sigma_{,+}^{2} du} v + u(x^2 + y^2) \\
\bar{x} &= ux \\
\bar{y} &= uy
\end{aligned}
$$

to get the following line element

$$(ds)^2 = d\bar{u}\, d\bar{v} - d\bar{x}^2 - d\bar{y}^2 - \frac{1}{2}\left(\frac{d\sigma}{du}\right)^2 (\bar{u}\bar{v} - \bar{x}^2 - \bar{y}^2)\, d\bar{u}^2$$

From here we see that if $d\sigma/du = 0$ for some region the region is flat.

Now we turn to the geodesic deviation formula to see how the scalar wave interacts with test particles. Let t^{μ} be the unit tangent of the worldline for an observer and ω^{μ} the space-like infinitesimal connecting vector, so we have

$$\frac{D^2\omega^{\mu}}{DS^2} = R^{\mu}_{\nu\sigma\tau} t^{\nu} t^{\tau} \omega^{\sigma}$$

Using expression (3.2) we get

$$\frac{D^2\omega^{\mu}}{DS^2} = -\frac{1}{2}\left(\frac{d\sigma}{ds}\right)^2 [(\hat{\xi}^{\mu}(\hat{\xi}\,.\,\omega) + \hat{\eta}^{\mu}(\hat{\eta}\,.\,\omega)] \tag{3.3}$$

where

$$
\begin{aligned}
\hat{\xi}_{\mu} &= \xi_{\mu} - \frac{\xi\,.\,t}{K\,.\,t} K_{\mu} \\
\eta_{\mu} &= \eta_{\mu} - \frac{\eta\,.\,t}{K\,.\,t} K_{\mu} \\
\frac{d\sigma}{ds} &= t^{\mu}\sigma_{,\mu}
\end{aligned} \tag{3.4}
$$

$\hat{\xi}_{\mu}$ and $\hat{\eta}_{\mu}$ are unit vectors orthogonal to t^{μ} and among themselves. We can think of them as defining the instantaneous plane of symmetry seen by the observer t^{μ}. This is consistent because K_{μ} is also a Killing vector and therefore any linear combination $\alpha\xi_{\mu} + \beta\eta_{\mu} + \gamma K_{\mu}$ is a motion for α, β and γ constants.

Now we notice that the covariant derivative of $\hat{\xi}^{\mu}$ and $\hat{\eta}^{\mu}$ along the observer worldline vanishes. This fact is essential if $\hat{\xi}^{\mu}$ and $\hat{\eta}^{\mu}$ are to be considered space-like unit vectors defining a local reference frame for the observer. To prove this statement it is convenient to use the following relations

$$K_{\mu;\nu} = 0$$

$$\xi^{(x)}_{\mu;\nu} = e^{-\omega}\frac{d\mu}{du}(\xi^{(x)}_\mu K_\nu - \xi^{(x)}_\nu K_\mu)$$

$$\eta^{(y)}_{\mu;\nu} = e^{-\omega}\frac{d\mu}{du}(\eta^{(y)}_\mu K_\nu - \eta^{(y)}_\nu K_\mu) \tag{3.5}$$

where $\xi^{(x)}_\mu$ and $\eta^{(y)}_\mu$ are the Killing vectors $\partial/\partial x$ and $\partial/\partial y$ respectively.

Multiplying (3.3) by $\hat{\xi}_\mu$ and $\hat{\eta}_\mu$ we obtain

$$\frac{d^2\omega\,.\,\hat{\xi}}{ds^2} + \frac{1}{2}\left(\frac{d\sigma}{ds}\right)^2 \omega\,.\,\hat{\xi} = 0$$

$$\frac{d^2\omega\,.\,\hat{\eta}}{ds^2} + \frac{1}{2}\left(\frac{d\sigma}{ds}\right)^2 \omega\,.\,\hat{\eta} = 0$$

$$\frac{d^2\omega\,.\,\hat{\delta}}{ds^2} = 0$$

where $\hat{\delta}^\mu$ is a space like unit vector orthogonal to t^μ, $\hat{\xi}^\mu$ and $\hat{\eta}^\mu$.

So if $\mathbf{r}$ is the infinitesimal position vector of a neighboring particle the observer sees an effective time-dependent potential acting on this particle given by

$$V(\mathbf{r}) = \frac{m}{4}\left(\frac{d\sigma}{ds}\right)^2 (\hat{K} \times \mathbf{r})^2$$

where m is the mass of the test particle and $\hat{K}$ a unit vector along the direction of incidence.

An entirely similar calculation gives us the effective potential for Brans–Dicke theory. The only difference is that particles travel along geodesics of a metric conformally related to (3.1); the conformal factor is given by (2.4). We just quote the result;

$$V(\mathbf{r}) = \frac{m}{4}\left[\frac{2\omega+4}{2\omega+3}\left(\frac{d\sigma}{ds}\right)^2 - \frac{1}{\sqrt{\omega+\frac{3}{2}}}\frac{d^2\sigma}{ds^2}\right](\hat{K} \times \mathbf{r})^2$$

σ is related to Brans–Dicke scalar field by

$$\phi = e^{\sigma/\sqrt{\omega+\frac{3}{2}}}$$

It is interesting that the force may become repulsive.

We shall consider now the following physical situation: two test particles initially at rest acquire a relative velocity after the wave packet passes by them. What is the final relative velocity?

This question has a definite meaning only because the particles find themselves initially and finally in a flat space-time.

We shall study this problem for a slightly more general space time defined by

$$ds^2 = e^{\omega(u)} du\, dv - e^{\mu(u)}(dx^2 + dy^2)$$

where $\omega(u)$ and $\mu(u)$ are not related.

An obvious coordinate transformation allows us to write this line element as

$$ds^2 = e^{\omega(\mu)} d\mu\, dv - e^{\mu}(dx^2 + dy^2) \tag{3.6}$$

and therefore makes the writing a little shorter.

The geodesics are given by

$$\begin{aligned}
\frac{d\mu}{ds} &= A e^{-\omega} \\
\frac{dv}{ds} &= \frac{1}{A} + \frac{1}{A} e^{-\mu}(a_x^2 + a_y^2) \\
\frac{dx}{ds} &= a_x e^{-\mu} \\
\frac{dy}{ds} &= a_y e^{-\mu}
\end{aligned} \tag{3.7}$$

where A, a_x and a_y are constants defining the initial unit tangent.

We shall also need the law of parallel transport from event $(u, 0, 0, 0)$ to event (u, v, x, y). The actual path is immaterial because the transport is to be carried out in flat space time only. We chose the path $(u, 0, 0, 0) \to (u, v, 0, 0) \to (u, v, x, 0) \to (u, v, x, y)$ along Killing directions. The law of parallel transport is given by

$$\begin{aligned}
\xi^+ &= \xi_0^{\,+} \\
\xi^- &= \xi_0^{\,-} - e^{\mu-\omega}(\xi_0^{\,x} x + \xi_0^{\,y} y) + \tfrac{1}{4} e^{\mu-\omega} \xi_0^{\,+}(x^2 + y^2) \\
\xi^x &= \xi_0^{\,x} - \tfrac{1}{2}\xi_0^{\,+} x \\
\xi^y &= \xi_0^{\,y} - \tfrac{1}{2}\xi_0^{\,+} y
\end{aligned} \tag{3.8}$$

where $\xi_0^{\,\mu}$ is the vector at $(u, 0, 0, 0)$ and ξ^μ is the parallel transport of $\xi_0^{\,\mu}$.

Without loss of generality we assume the wave packet to be located within $0 \leqslant \mu \leqslant \bar{\mu}$, and $\omega(0) = 0$. From (3.8) we conclude that if the wordline of particle 1 is defined by (A, a_x, a_y) and hits event $(u, 0, 0, 0)$ then particle's 2 worldline initially at rest with respect to 1, is defined by $(A, a_x - \tfrac{1}{2}Ax, a_y - \tfrac{1}{2}Ay)$ and hits (u, v, x, y).

From now on it is a matter of simple computation to get the final relative velocity.

The result is:

$$\frac{1}{\sqrt{1 - V_R^{\,2}/c^2}} = 1 + \frac{1}{8}\frac{d^2}{s^2}\left[e^{(1/2)\bar{\mu} - \omega(\bar{\mu})} \int_0^{\bar{\mu}} e^{\omega} d\mu \int_0^{\bar{\mu}} e^{-\mu/2} \frac{d}{d\mu} e^{\omega - \mu/2} d\mu \right]^2$$

where s is the proper time for the wave to pass by the particles (same for both). **d** is defined as follows: $\hat{\xi}_\mu$ and $\hat{\eta}_\mu$ defined by (3.4) generates a plane of symmetry (wave front) for observer t^μ. For the initial portion of space time we define a parallel vector field $t^\mu(x)$ by transporting t^μ at $(u, 0, 0, 0)$ all over it. Using (3.5) we get

$$\hat{\xi}_{\mu;\nu} = \hat{\eta}_{\mu;\nu} = 0$$

Now we define a unit vector field $\delta^\mu(x)$ to be orthogonal to t^μ, ξ^μ and η^μ. Obviously $\hat{\delta}_{\mu;\nu} = 0$ too. These vectors define a Lorentz frame over the initial flat space time. **d** is the projection of the three vector joining particles 1 and 2 upon the plane of symmetry. This result applied to the mass zero scalar field gives

$$\frac{1}{\sqrt{(1 - V_R{}^2/c^2)}} = 1 + \frac{1}{8}\frac{\mathbf{d}^2}{s^2}\left[\exp\left[\int_0^{\bar{\mu}} \left(\frac{d\sigma}{d\mu}\right)^2 d\mu\right] \int_0^{\bar{\mu}} \exp\left[\frac{\mu}{2}\right] \exp\left[-\int_0^{\bar{\mu}} \left(\frac{d\sigma}{d\mu}\right)^2 d\mu\right]\right.$$

$$\left. \times \int_0^{\bar{\mu}} \exp\left[-\frac{\mu}{2}\right] \frac{d}{d\mu} \exp\left[-\int_0^{\mu} \left(\frac{d\sigma}{d\mu}\right)^2 d\mu^2\right]\right]^2$$

For Brans-Dicke field we get

$$\frac{1}{\sqrt{(1 - V_R{}^2/c^2)}} = 1 + \frac{1}{8}\frac{\mathbf{d}^2}{s^2}\left[\exp\left[-\frac{\lambda\sigma}{2}\right] \exp\left[\int_0^{\mu} \left(\frac{d\sigma}{d\mu}\right)^2 d\mu\right] \int_0^{\mu} \exp\left[\frac{\mu}{2} + \lambda\sigma\right]\right.$$

$$\left. \times \exp -\int_0^{\mu} \left(\frac{d\sigma}{d\mu}\right)^2 d\mu \int_0^{\mu} \exp\left[-\frac{\mu}{2}\right] \frac{d}{d\mu} \exp\left[-\int_0^{\mu} \left(\frac{d\sigma}{d\mu}\right)^2 d\mu^2\right]\right]^2$$

where $\sigma(0) = 0$, $\sigma(\bar{\mu}) = \bar{\sigma}$ and $\lambda = -(1/\sqrt{(\omega + \frac{3}{2})})$.

References

Dicke, R. H. (1962). *Physical Review*, **125**, 2163.
Taub, A. H. (1956). *Physical Review*, **103**, 454.